Student Number:
Teacher:

St George Girls High School

Mathematics Advanced

2023 Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the Multiple-Choice answer sheet provided

For questions in **Section II:**

- Answer the questions in the booklets provided
- Show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks
Booklet 1 (pages 8-20)
Booklet 2 (pages 24-38)

- Attempt Questions 11 – 32
- Allow about 2 hours and 45 minutes for this section

Q1 – Q10	/10
Q11 – Q14	/14
Q15 – Q17	/12
Q18 – Q20	/10
Q21 – Q23	/15
Q24 – Q27	/12
Q28 – Q30	/15
Q31 – Q32	/12
Total	**/100**
	%

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. What is the gradient of the linear relationship below?

$$3x - 5y + 30 = 0$$

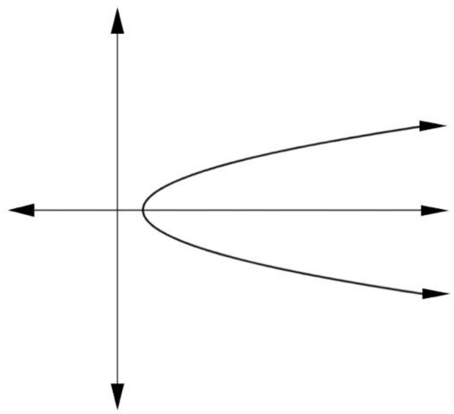
(A) $\frac{3}{5}$

(B) $-\frac{3}{5}$

(C) -3

(D) -5

2. What type of relation is shown below?



(A) one-to-one

(B) many-to-many

(C) many-to-one

(D) one-to-many

3. A six sided dice is rolled. Which of the following events are mutually exclusive?

(A) Rolling a four or an even number

(B) Rolling a number less than 3 or an odd number

(C) Rolling a five or an odd number

(D) Rolling a two or a number greater than 4

4. If $f(x) = e^{\sin x}$, what is the value of $f'(0)$?

(A) 1

(B) 0

(C) -1

(D) e

5. A pack of playing cards consists of four suits (Hearts, Diamonds, Clubs and Spades) containing thirteen cards each (Ace, 2 to 10, Jack, Queen and King). The pack of cards is shuffled and then a card is drawn at random. Given it is red, what is the probability that it is a queen or a diamond?

(A) $\frac{7}{13}$

(B) $\frac{1}{2}$

(C) $\frac{4}{13}$

(D) $\frac{17}{52}$

6. The primitive of $y = 5^{2x}$ is:

(A) $\frac{5^{2x}}{\ln 5} + c$.

(B) $5^{2x} \ln 5 + c$.

(C) $\frac{5^{2x}}{2\ln 5} + c$.

(D) $5.5^{2x} \ln 5 + c$.

7. The first three terms of an arithmetic series are 3, 7 and 11. If the last term is 75, find the sum of the series.

(A) 17

(B) 19

(C) 739

(D) 741

8. The height of the tide in a harbour can be modelled using the sine function. The time, t in hours, for the tide to complete one full cycle from high tide to low tide and back to high tide is 12 hours. Which of the following could be the function representing the height of the tide?

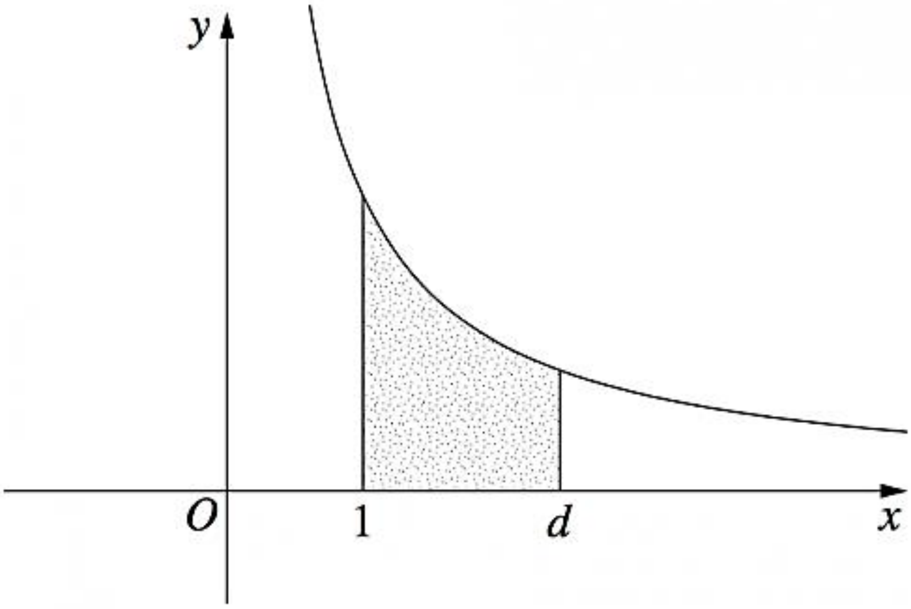
(A) $h = \sin(\frac{\pi t}{3})$

(B) $h = \sin(\frac{\pi t}{6})$

(C) $h = \sin(\frac{\pi t}{12})$

(D) $h = \sin(\frac{\pi t}{18})$

9. The diagram below shows the area under the curve $y = \frac{3}{x}$ from $x = 1$ to $x = d$.



What value of d makes the shaded area equal to 6?

(A) $2e$

(B) e

(C) $2e^2$

(D) e^2

10. Which of the following is true?

(A) $\int_0^1 e^{-x}\,dx < \int_1^2 e^{-x}\,dx$

(B) $\int_0^1 e^{x}\,dx < \int_{-1}^0 e^{x}\,dx$

(C) $\int_0^1 e^{-x}\,dx > \int_1^2 e^{-x}\,dx$

(D) $\int_0^1 e^{x}\,dx > \int_1^2 e^{x}\,dx$

END OF SECTION I

Question 11 (4 marks)

Consider the discrete probability distribution below.

x	1	2	3	4
$P(X = x)$	$2k$	0.15	0.25	k

(a) Find $E(X)$. **2**

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(b) Find the variance. **2**

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Question 12 (2 marks)

For what values of k does the quadratic equation $3x^2 - 2x + (4k - 12) = 0$ have real roots? **2**

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Question 13 (3 marks)

Find the gradient of the normal to $y = x^3 \cos x$ at the point $(\frac{\pi}{2}, 0)$. **3**

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Question 14 (5 marks)

Find:

(a) $\int 8e^{4x+1}\,dx$. **1**

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(b) $\int \frac{x^3-2x^2+3x}{x}\,dx$. **2**

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(c) $\int \frac{x^2}{2x^3-7}\,dx$. **2**

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Question 15 (6 marks)

The displacement of a particle is given by the formula $x = t^3 - 27t$ where x is in metres and t is in seconds.

(a) Find the acceleration of the particle at 3 seconds. **2**

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(b) Find when the particle is stationary. **2**

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(c) Sketch the graph of the velocity against time below. **2**

Question 16 (3 marks)

Given that $f'(x) = \sin 2x$ and that $f\left(\frac{\pi}{8}\right) = 0$, find $f(x)$. **3**

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Question 17 (3 marks)

If the second term of a geometric series is 12 and the seventh term is 2916, find the sum of the first 5 terms. **3**

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Question 18 (4 marks)

Town A is 585km from Town B on a bearing of $320°T$. Town C is 940km due north of Town A.

(a) Draw a diagram representing the information above. **1**

(b) Calculate the distance from Town C to Town B. (Answer to the nearest kilometre). **3**

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Question 19 (3 marks)

The circle $x^2 + y^2 - 6x + 8y - 11 = 0$ is translated to the left by 4 units and up by 3 units. **3**

What is the centre and radius of the new circle?

Question 20 (3 marks)

Solve $2\ln x = \ln(2x + 3)$. **3**

Question 21 (10 marks)

Let $f(x) = \frac{1}{4}(x-2)^2(x+1)$.

(a) Show that $f'(x) = \frac{1}{4}(3x^2 - 6x)$. **2**

(b) Find the coordinates of the stationary points of $y = f(x)$ and determine their nature. **2**

(c) Find the coordinates of any point(s) of inflection. **2**

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(d) Sketch the graph of $y = f(x)$ over the domain $[-2, 4]$, showing all important features. **3**

(e) Determine the global maximum of this function within the domain $[-2, 4]$. **1**

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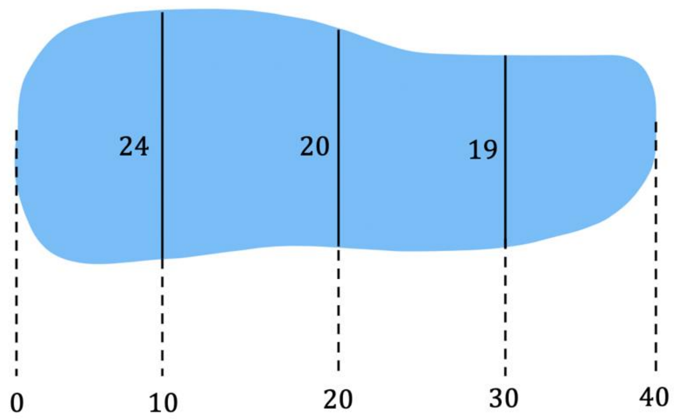
Question 22 (3 marks)

The diagram below shows the width of a lake at 10 metre intervals. Use the trapezoidal rule to estimate the surface area of the water. **3**



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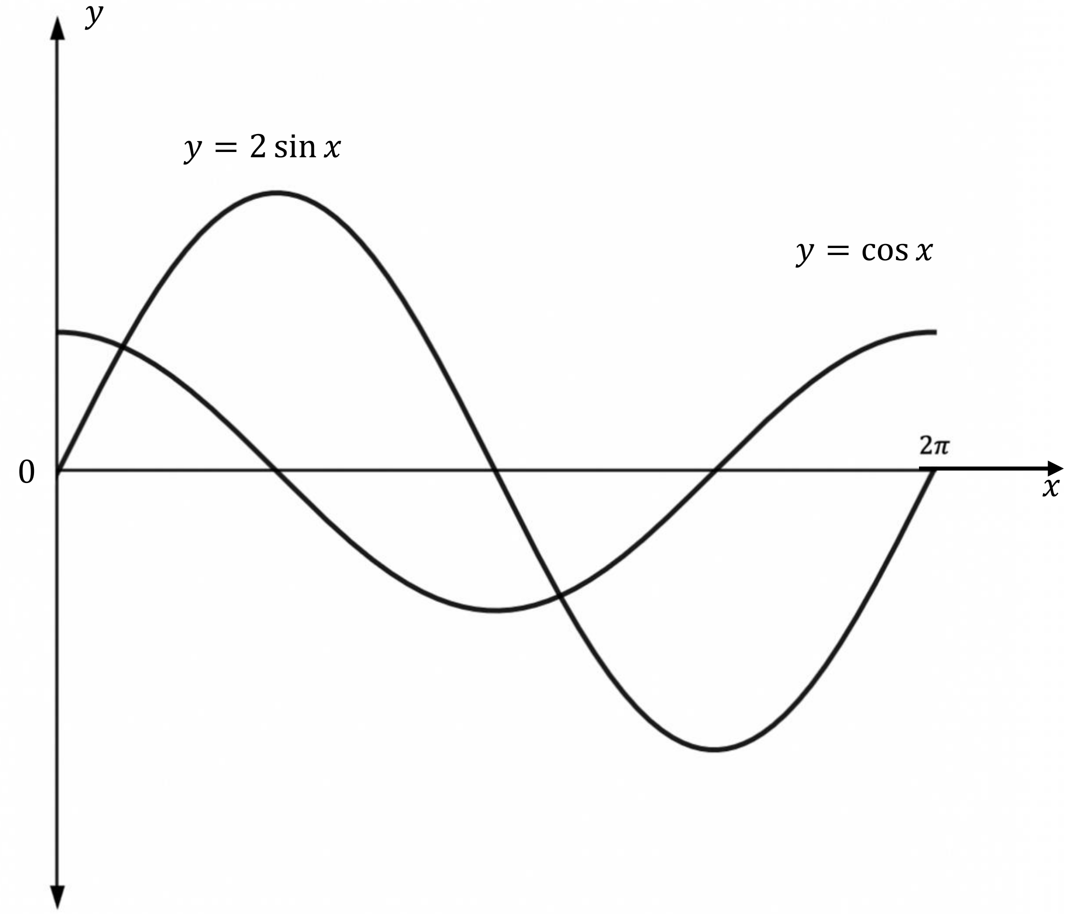
Question 23 (2 marks) **2**

The diagram below shows the graphs of the curves $y = 2\sin x$ and $y = \cos x$ for $0 \le x \le 2\pi$. State the translation and dilation required to transform the graph of the curve $y = \cos x$ into the graph of the curve $y = 2\sin x$.



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End of Booklet 1

Proceed to Booklet 2 for Questions 24-32

Question 24 (3 marks)

History and Geography are two of the subjects students may decide to study. For a group of 40 students, the following is known.

- 7 students study neither History nor Geography
- 20 students study History
- 18 students study Geography

(a) A student is chosen at random. By drawing a Venn diagram, or otherwise, find the probability that the student studies both History and Geography. **2**

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(b) A student is chosen at random. Given that the student studies Geography, what is the probability that the student does NOT study History? **1**

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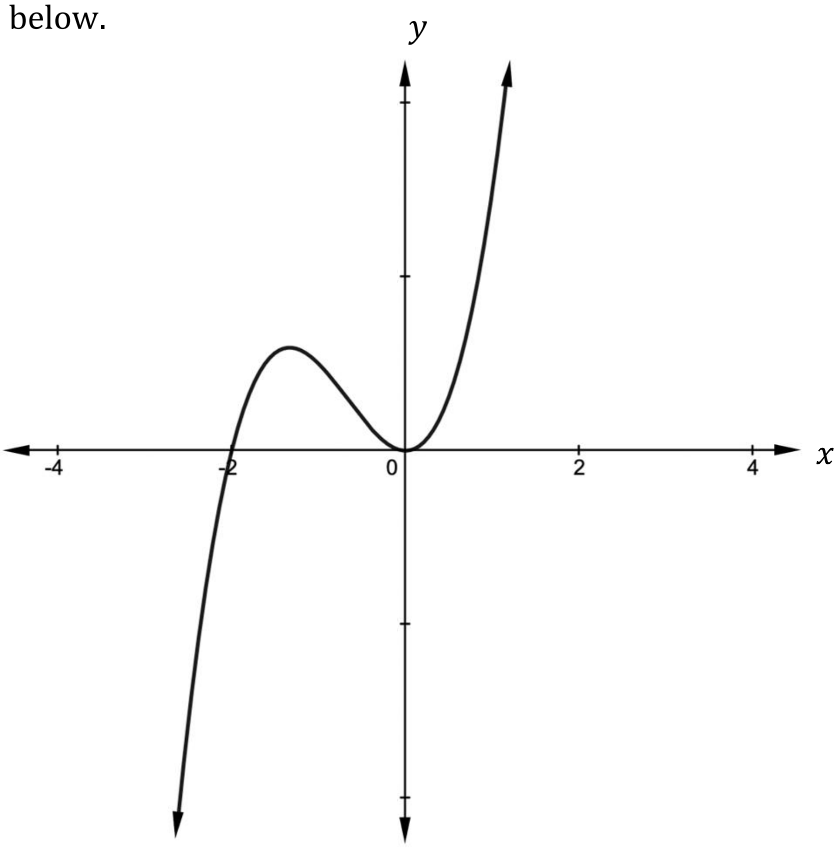
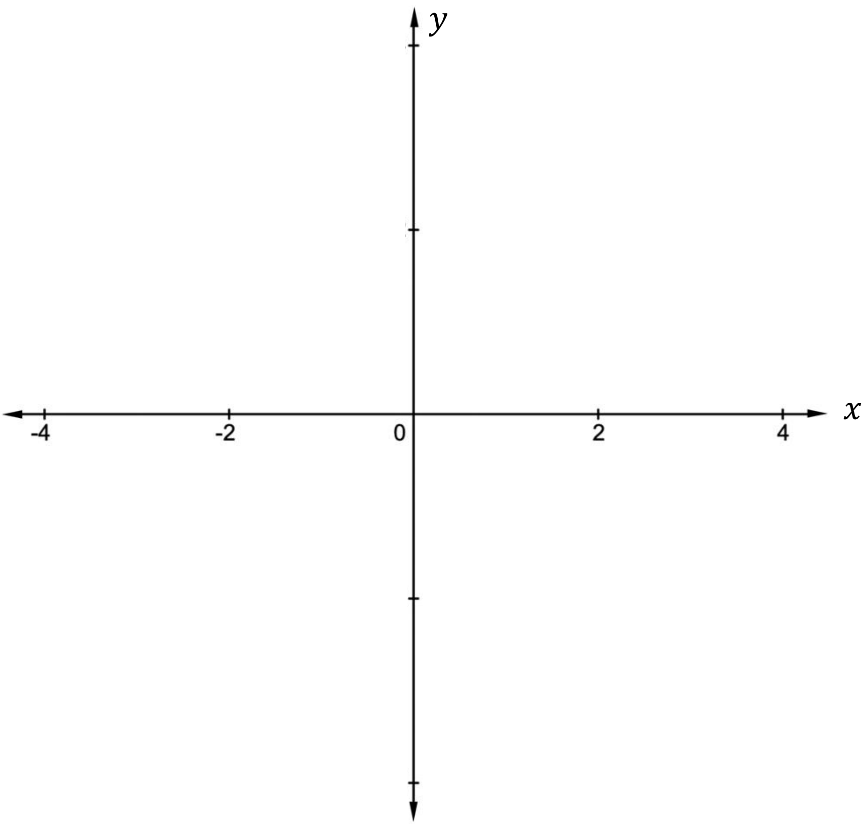
Question 25 (2 marks)

Consider the curve below.



Sketch the graph of the derivative in the space provided below: **2**



Question 26 (4 marks)

The limiting sum of an infinite geometric series is five times the first term.

(a) Show that the common ratio of the series is 0.8. **2**

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(b) If the first term is 20, find the smallest value of n for which the n^{th} term is less than 1. **2**

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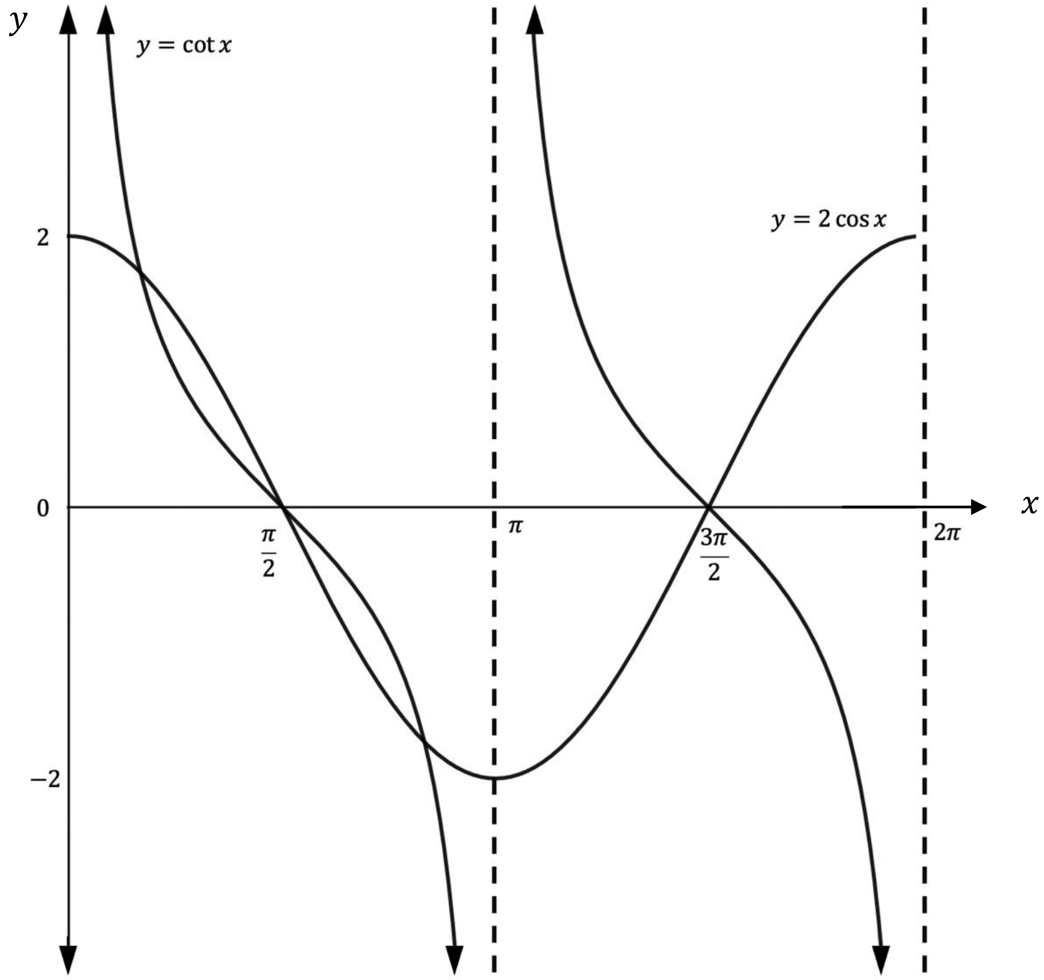
Question 27 (3 marks)

(a) Prove that $\frac{d}{dx}(xe^x) = xe^x + e^x$. **1**

(b) Hence, find $\int xe^x \, dx$. **2**

Question 28 (7 marks)



The diagram shows the graphs of the functions $y = 2\cos x$ and $y = \cot x$ for $0 \leq x \leq 2\pi$.

(a) Write down the periods of each of the functions $y = 2\cos x$ and $y = \cot x$. **2**

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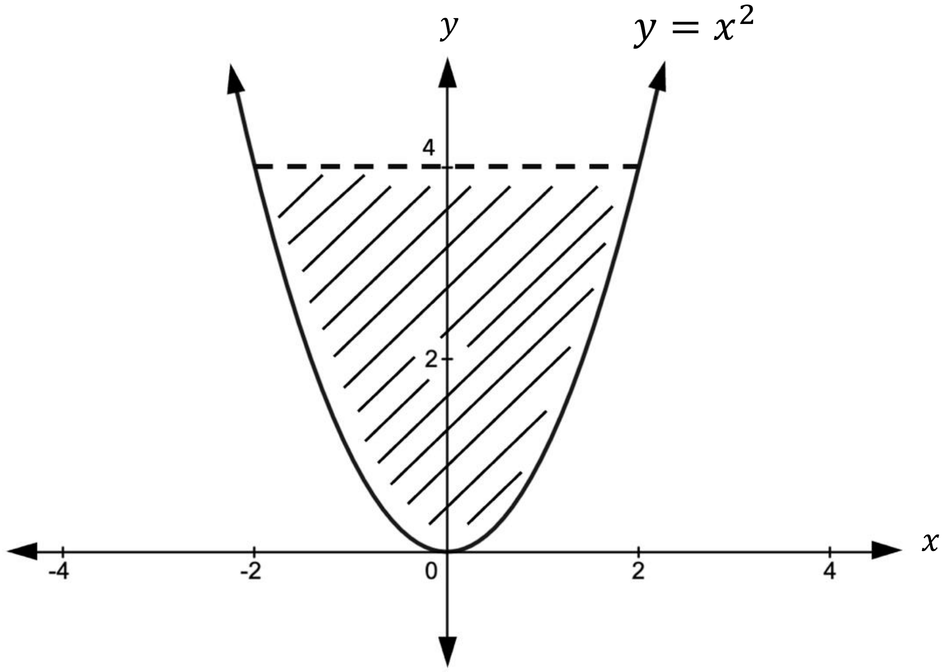
(b) Show that the x −coordinates of the points of intersections of the two curves are the solutions of the equation $\cos x\,(2\sin x - 1) = 0$ for $0 \leq x \leq 2\pi$, then solve this equation. **3**

(c) Find in simplest exact form the total area of the bounded region enclosed by the two curves for $0 \leq x \leq \pi$. **3**

Question 29 (3 marks) **3**

Find the area of the shaded region below:



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Question 30 (4 marks)

A population, P, which is initially 5000, varies according to the formula $P = 5000b^{-\frac{t}{10}}$, where b is a positive constant and t is time in years, $t \geq 0$. **4**

The population is 1250 after 20 years. Find the value of t, correct to one decimal place, for which the instantaneous rate of decrease is 30 people per year.

Question 31 (5 marks)

(a) Show that $\frac{d}{dx}[\log_4(\tan x)] = \frac{1+\tan^2 x}{\ln 4 \tan x}$. **3**

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(b) Hence, find the angle of inclination to $y = \log_4(\tan x)$ at $x = \frac{\pi}{4}$ to the nearest degree. **2**

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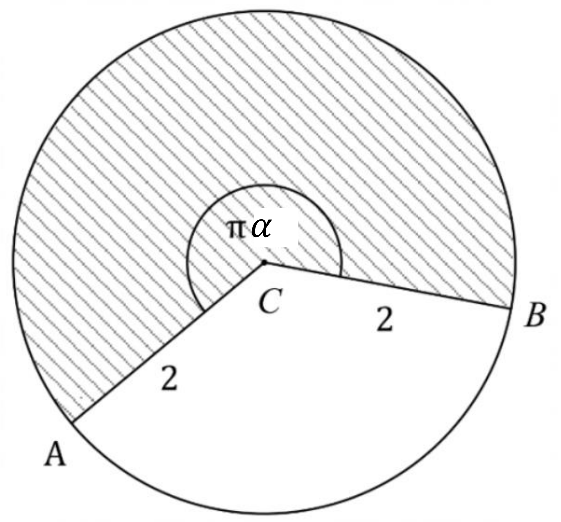
Question 32 (7 marks)



The reflex angle at the centre C of a circle of radius 2cm is $\pi\alpha$ radians, $0 < \alpha < 2$, as shown in the diagram above.

(a) Find the length of the arc of the shaded sector. **1**

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(b) The shaded sector is cut from the circle along the radii CA and CB and folded to make a cone. Find the radius of the cone, in terms of α. **1**

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Student Number:
Teacher:

Solutions

St George Girls High School

Mathematics Advanced

2023 Trial HSC Examination

Solutions

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the Multiple-Choice answer sheet provided

For questions in **Section II:**

- Answer the questions in the booklets provided
- Show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 88 marks (pages 11 – 36)

- Attempt Questions 11 – 32
- Allow about 2 hours and 45 minutes for this section

Q1 – Q10	/10
Q11 – Q14	/14
Q15 – Q17	/12
Q18 – Q20	/9
Q21 – Q23	/15
Q24 – Q27	/12
Q28 – Q30	/14
Q31 – Q32	/12
Total	**/98**
	%

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. What is the gradient of the linear relationship below?

$$3x - 5y + 30 = 0$$

(A) $\frac{3}{5}$

(B) $-\frac{3}{5}$

(C) -3

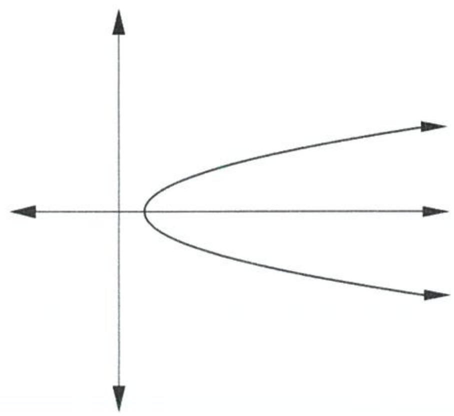
(D) -5

$5y = 3x + 30$

$y = \frac{3}{5}x + 6$

$m = \frac{3}{5}$

2. What type of relation is shown below?



(A) one-to-one

(B) many-to-many

(C) many-to-one

(D) one-to-many

3. A six sided dice is rolled. Which of the following events are mutually exclusive?

(A) Rolling a four or an even number

(B) Rolling a number less than 3 or an odd number

(C) Rolling a five or an odd number

(D) Rolling a two or a number greater than 4

4. If $f(x) = e^{\sin x}$ what is the value of $f'(0)$?

(A) 1

(B) 0

(C) -1

(D) e

$f'(x) = \cos x\, e^{\sin x}$

$f'(0) = \cos 0\, e^{\sin 0}$

$= 1$

5. A pack of playing cards consists of four suits (Hearts, Diamonds, Clubs and Spades) containing thirteen cards each (Ace, 2 to 10, Jack, Queen and King). The pack of cards is shuffled and then a card is drawn at random. Given it is red, what is the probability that it is a queen or a diamond?

(A) $\frac{7}{13}$

(B) $\frac{1}{2}$

(C) $\frac{4}{13}$

(D) $\frac{17}{52}$

6. The primitive of $y = 5^{2x}$ is:

(A) $\frac{5^{2x}}{\ln 5} + c$.

(B) $5^{2x} \ln 5 + c$.

(C) $\frac{5^{2x}}{2\ln 5} + c$.

(D) $5.5^{2x} \ln 5 + c$.

7. The first three terms of an arithmetic series are 3, 7 and 11. If the last term is 75, find the sum of the series.

(A) 17

(B) 19

(C) 739

(D) 741

8. The height of the tide in a harbour can be modelled using the sine function. The time, t in hours, for the tide to complete one full cycle from high tide to low tide and back to high tide is 12 hours. Which of the following could be the function representing the height of the tide?

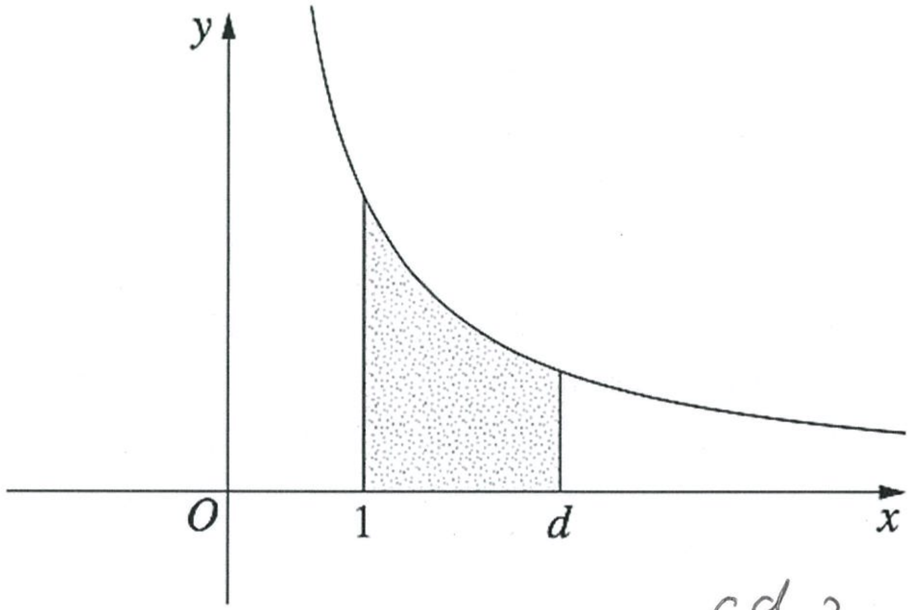
(A) $h = \sin(\frac{\pi t}{3})$

(B) $h = \sin(\frac{\pi t}{6})$

(C) $h = \sin(\frac{\pi t}{12})$

(D) $h = \sin(\frac{\pi t}{18})$

9. The diagram below shows the area under the curve $y = \frac{3}{x}$ from $x = 1$ to $x = d$.



What value of d makes the shaded area equal to 6?

(A) $2e$

(B) e

(C) $2e^2$

(D) e^2

$\int_1^d \frac{3}{x}\,dx = 6$

$3\int_1^d \frac{1}{x}\,dx = 6$

$3\times[\ln x]_1^d = 6$

$3\times(\ln d - \ln 1) = 6$

$3\ln d = 6$

$\ln d = 2$

$e^2 = d$

10. Which of the following is true?

(A) $\int_0^1 e^{-x}\,dx < \int_1^2 e^{-x}\,dx$

(B) $\int_0^1 e^{x}\,dx < \int_{-1}^0 e^{x}\,dx$

(C) $\int_0^1 e^{-x}\,dx > \int_1^2 e^{-x}\,dx$

(D) $\int_0^1 e^{x}\,dx > \int_1^2 e^{x}\,dx$

END OF SECTION I

Mathematics Advanced

Section II Answer Booklet 1

Student Number:
Teacher:

Section II
90 marks
Attempt Questions 11 – 32
Allow about 2 hours and 45 minutes for this section

Booklet 1 – Attempt Question 11 – 23 (50 marks)
Booklet 2 – Attempt Question 24 – 32 (38 marks)

Instructions

- Write your Teacher's Name and Student Number at the top of this page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided on page 22 of Booklet 1. If you use this space, clearly indicate which question you are answering.

Please turn over

Question 11 (4 marks)

Consider the discrete probability distribution below

x	1	2	3	4
$P(X = x)$	$2k$	0.15	0.25	k

(a) Find $E(X)$. **2**

$2k + 0.15 + 0.25 + k = 1$

$3k + 0.4 = 1$

$3k = 0.6$ Hence, $k = 0.2$

$E(X) = \sum x\, p(x)$

$= (1 \times 0.4) + (2 \times 0.15) + (3 \times 0.25) + (4 \times 0.2)$

$= 2.25$

(b) Find the variance. **2**

$Var(X) = \sum x^2 p(x) - \mu^2$

$= (1 \times 0.4) + (4 \times 0.15) + (9 \times 0.25) + (16 \times 0.2) - 2.25^2$

$= 1.3875$

Question 12 (2 marks)

For what values of k does the quadratic equation $3x^2 - 2x + (4k - 12) = 0$ have real roots? **2**

Real roots when $\Delta \geq 0$

$\Delta = b^2 - 4ac$

$\therefore b^2 - 4ac \geq 0$

$(-2)^2 - 4(3)(4K-12) \geq 0$

$4 - 12(4K-12) \geq 0$

$-48K + 148 \geq 0$

$-48K \geq -148$

$K \leq \frac{37}{12}$

Question 13 (3 marks)

Find the gradient of the normal to $y = x^3 \cos x$ at the point $(\frac{\pi}{2}, 0)$. **3**

$\frac{dy}{dx} = u'v + v'u$

$m_{tangent} = 3x^2 \cos x - x^3 \sin x$

$= 3(\frac{\pi}{2})^2 \cos(\frac{\pi}{2}) - (\frac{\pi}{2})^3 \sin(\frac{\pi}{2})$

$= \frac{3\pi^2}{4}(0) - \frac{\pi^3}{8}(1)$

$= -\frac{\pi^3}{8}$

$m_{normal} = \frac{8}{\pi^3}$

Question 14 (5 marks)

Find:

(a) $\int e^{4x+1}\,dx$. 1

$= \frac{e^{4x+1}}{4} + c$

(b) $\int \frac{x^3-2x^2+3x}{x}\,dx$. 2

$= \int \frac{x^3}{x} - \frac{2x^2}{x} + \frac{3x}{x}\,dx$

$= \int x^2 - 2x + 3\,dx$

$= \frac{x^3}{3} - x^2 + 3x + c$

(c) $\int \frac{x^2}{2x^3-7}\,dx$. 2

$= \frac{1}{6}\int \frac{6x^2}{2x^3-7}\,dx$

$= \frac{1}{6}\ln|2x^3-7| + c$

$= \frac{\ln|2x^3-7|}{6} + c$

Question 15 (6 marks)

The displacement of a particle is given by the formula $x = t^3 - 27t$ where x is in metres and t is in seconds.

(a) Find the acceleration of the particle after 3 seconds. **2**

$x = t^3 - 27t$

$\dot{x} = 3t^2 - 27$

$\ddot{x} = 6t$

when $t = 3$, $\ddot{x} = 18 m/s^2$

(b) Find when the particle is stationary. **2**

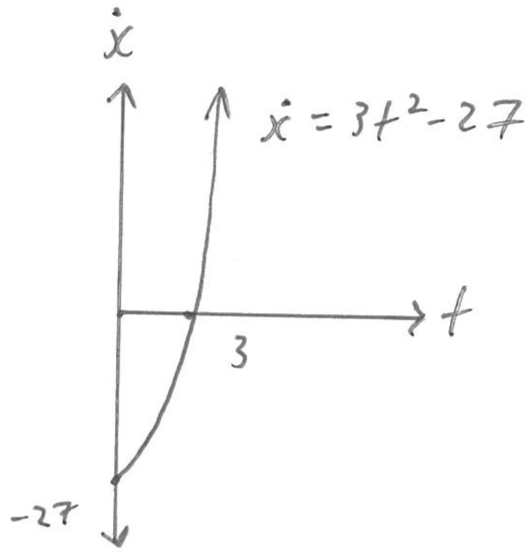
Stationary when $\dot{x} = 0$

$3t^2 - 27 = 0$

$t^2 = 9$

$t = 3 \quad t \geq 0$

(c) Sketch the graph of the velocity against time below. **2**



Question 16 (3 marks)

Given that $f'(x) = \sin 2x$ and that $f\left(\frac{\pi}{8}\right) = 0$, find $f(x)$. **3**

$f(x) = \int \sin 2x \, dx$

$f(x) = -\frac{1}{2}\cos 2x + c$

$0 = -\frac{1}{2}\cos\left(2 \times \frac{\pi}{8}\right) + c$

$0 = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right) + c$

$0 = -\frac{1}{2} \times \frac{1}{\sqrt{2}} + c$

$c = \frac{1}{2\sqrt{2}}$

$\therefore f(x) = -\frac{1}{2}\cos 2x + \frac{1}{2\sqrt{2}}$

Question 17 (3 marks)

If the second term of a geometric series is 12 and the seventh term is 2916, find the sum of the first 5 terms. **3**

① $12 = ar$

② $2916 = ar^6$

② ÷ ①

$243 = r^5$

$r = 3$

$\therefore a = 4$

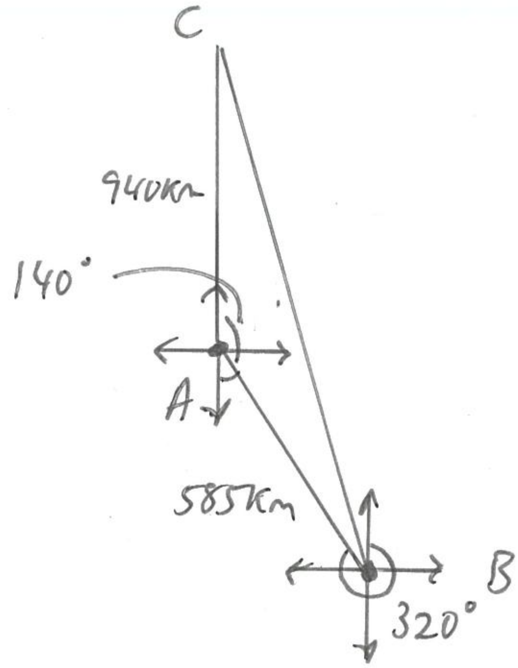
Now, $S_5 = \frac{4(3^5 - 1)}{2}$

$= 484$

Question 18 (3 marks)

Town A is 585km from Town B on a bearing of $320°T$. Town C is 940km due north of Town A. **3**

Calculate the distance from Town C to Town B. (Answer to the nearest kilometre).



$\angle CAB = 140°$ (alternate angles)

$CB^2 = 585^2 + 940^2 - 2 \times 585 \times 940 \times \cos 140$

$CB = 1438$ Km

Question 19 (3 marks)

The circle $x^2 + y^2 - 6x + 8y - 11 = 0$ is translated to the left by 4 units and up by 3 units. **3**

What is the centre and radius of the new circle?

$x^2 - 6x + y^2 + 8y = 11$

$x^2 - 6x + \left(-\frac{6}{2}\right)^2 + y^2 + 8y + \left(\frac{8}{2}\right)^2 = 11 + 9 + 16$

$(x-3)^2 + (y+4)^2 = 36$

translate left 4 units, up 3 units

$(x+1)^2 + (y+1)^2 = 36$

centre $(-1, -1)$ radius $= 6$

Question 20 (3 marks)

Solve $2 \ln x = \ln(2x + 3)$. **3**

$\ln x^2 = \ln(2x+3)$

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

~~$x = -1$~~ or $x = 3$

Disregard

$\therefore x = 3$

Question 21 (10 marks)

Let $f(x) = \frac{1}{4}(x-2)^2(x+1)$.

(a) Show that $f'(x) = \frac{1}{4}(3x^2 - 6x)$. **2**

$f'(x) = u'v + v'u$

$= \frac{1}{4}\left[2(x-2)(x+1) + (x-2)^2\right]$

$= \frac{1}{4}(x-2)(2x+2+x-2)$

$= \frac{1}{4}(x-2)(3x)$

$= \frac{1}{4}(3x^2-6x)$ as required

(b) Find the coordinates of the stationary points of $y = f(x)$ and determine their nature. **2**

Stationary points when $f'(x) = 0$

$\frac{3x}{4}(x-2) = 0$

$\therefore$ Stationary points at $(0,1)$ and $(2,0)$

x	-1	0	1	2	3
$\frac{dy}{dx}$	$\frac{9}{4}$	0	$-\frac{3}{4}$	0	$\frac{9}{4}$
Slope	/	–	\	–	/

$\therefore$ $(0,1)$ is a maximum

$(2,0)$ is a minimum

(c) Find the coordinates of any point(s) of inflection. **2**

points of inflection when $f''(x) = 0$

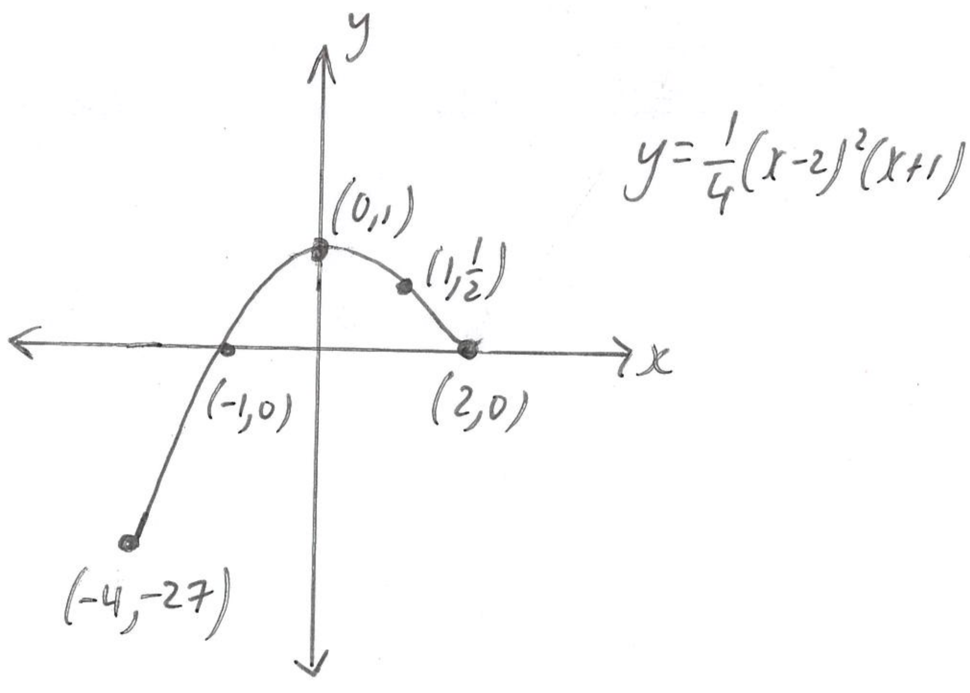
$$f'(x) = \frac{3x^2}{4} - \frac{3x}{2}$$

$$f''(x) = \frac{3x}{2} - \frac{3}{2}$$

$$0 = \frac{3}{2}(x-1)$$

∴ non-horizontal point of inflexion at $\left(1, \frac{1}{2}\right)$ as there is no stat. pnt. at $x=1$

(d) Sketch the graph of $y = f(x)$ over the domain $[-4,2]$, showing all important features. **3**

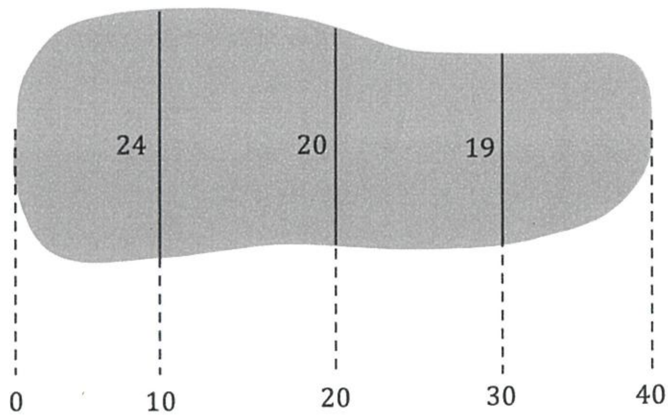


(e) Determine the global maximum of this function within the domain [−4,2]. **1**

$y = 1$

Question 22 (3 marks)

The diagram below shows the width of a lake at 10 metre intervals. Use the trapezoidal rule to estimate the surface area of the water. **3**



x	0	10	20	30	40
y	0	24	20	19	0

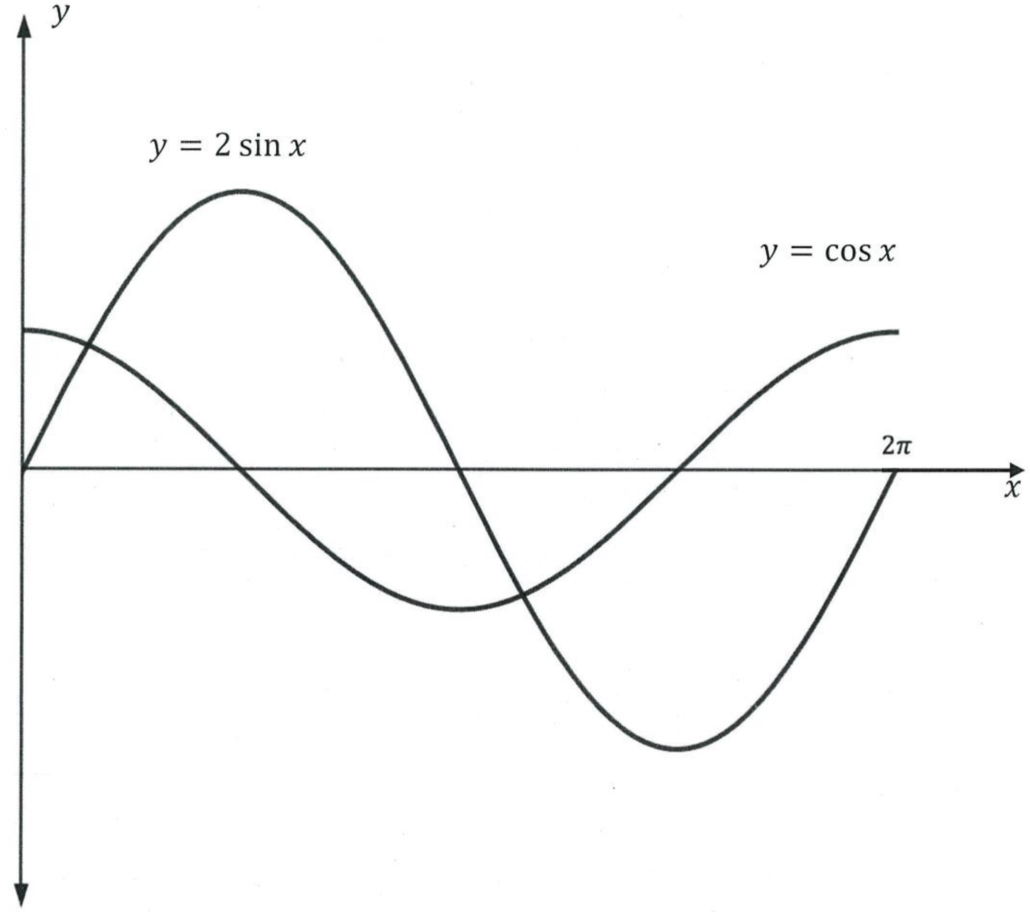
Approx area $= \frac{10}{2}\left(0 + 2(24+20+19) + 0\right)$

$= 5(126)$

$= 630$ units squared

Question 23 (2 marks) **2**

The diagram below shows the graphs of the curves $y = 2\sin x$ and $y = \cos x$ for $0 \le x \le 2\pi$. State the translation and dilation required to transform the graph of the curve $y = \cos x$ into the graph of the curve $y = 2\sin x$.



vertical dilation by a factor of 2
translation right by $\frac{\pi}{2}$ units

End of Booklet 1

Mathematics Advanced

Section II Answer Booklet 2

Student Number:
Teacher:

Section II

Booklet 2 – Attempt Question 24 – 32 (38 marks)

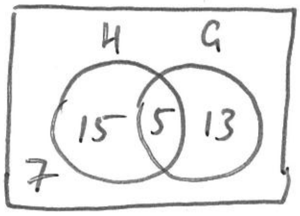
Instructions

- Write your Teacher's Name and Student Number at the top of this page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided on pages 37–38 of Booklet 2. If you use this space, clearly indicate which question you are answering.

Question 24 (3 marks)

History and Geography are two of the subjects students may decide to study. For a group of 40 students, the following is known.

- 7 students study neither History nor Geography
- 20 students study History
- 18 students study Geography



(a) A student is chosen at random. By drawing a Venn diagram, or otherwise, find the probability that the student studies both History and Geography **2**

$= \frac{5}{40}$

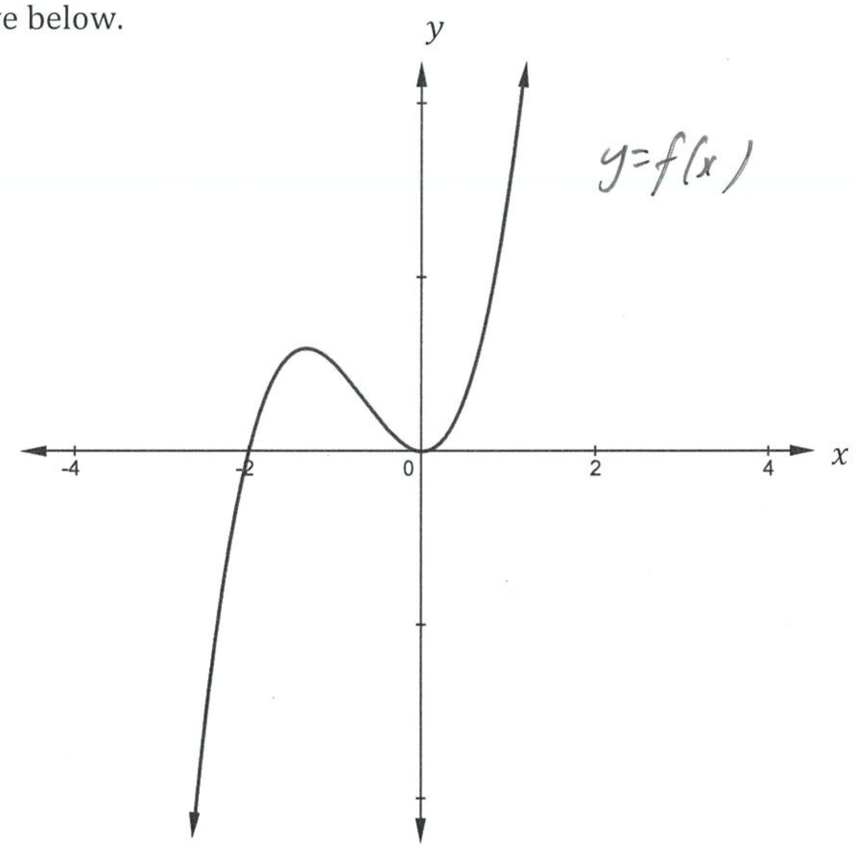
$= \frac{1}{8}$

(b) A student is chosen at random. Given that the student studies Geography, what is the probability that the student does NOT study History? **1**

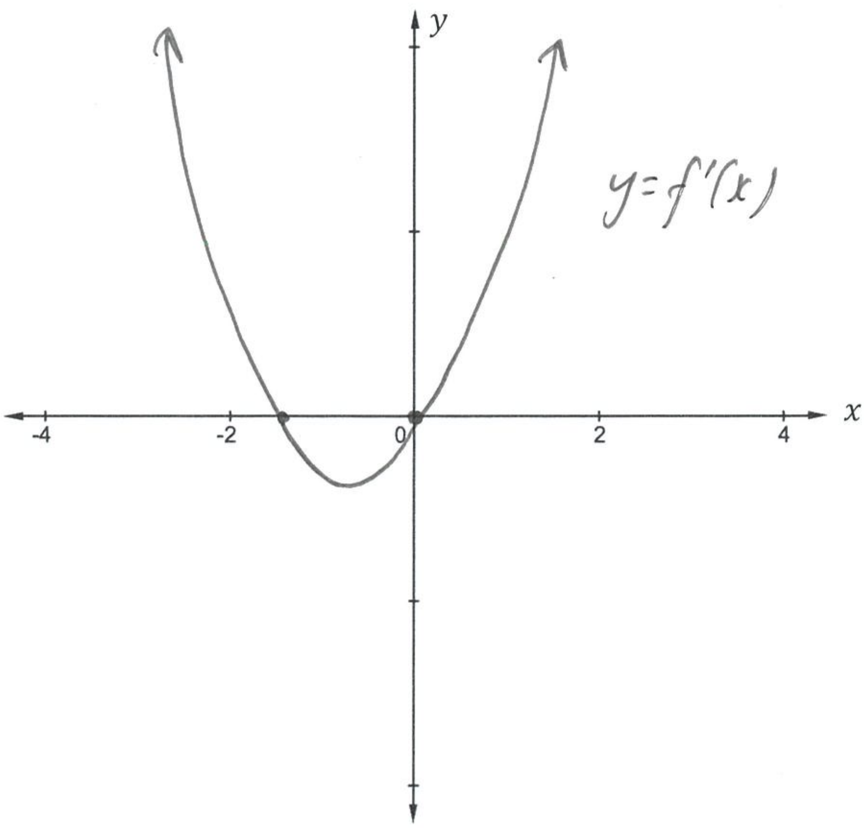
$= \frac{13}{18}$

Question 25 (2 marks)

Consider the curve below.



Sketch the graph of the derivative in the space provided below: **2**



Question 26 (4 marks)

The limiting sum of an infinite geometric series is five times the first term.

(a) Show that the common ratio of the series is 0.8. **2**

$S_\infty = \frac{a}{1-r}$

$5a = \frac{a}{1-r}$

$5a(1-r) = a$

$1-r = \frac{1}{5}$

$r = \frac{4}{5} = 0.8$ as required

(b) If the first term is 20, find the smallest value of n for which the n^{th} term is less than 1. **2**

$t_n = ar^{n-1}$

$20(0.8)^{n-1} < 1$

$0.8^{n-1} < \frac{1}{20}$

$(n-1)\ln(0.8) < \ln\left(\frac{1}{20}\right)$

$n-1 > \frac{\ln(0.05)}{\ln(0.8)}$

$n > 14.4$

15

Question 27 (3 marks)

(a) Prove that $\frac{d}{dx}(xe^x) = xe^x + e^x$. **1**

$\frac{d}{dx}xe^x = u'v + v'u$

$= e^x + xe^x$

$= xe^x + e^x$ as required

(b) Hence find $\int xe^x \, dx$. **2**

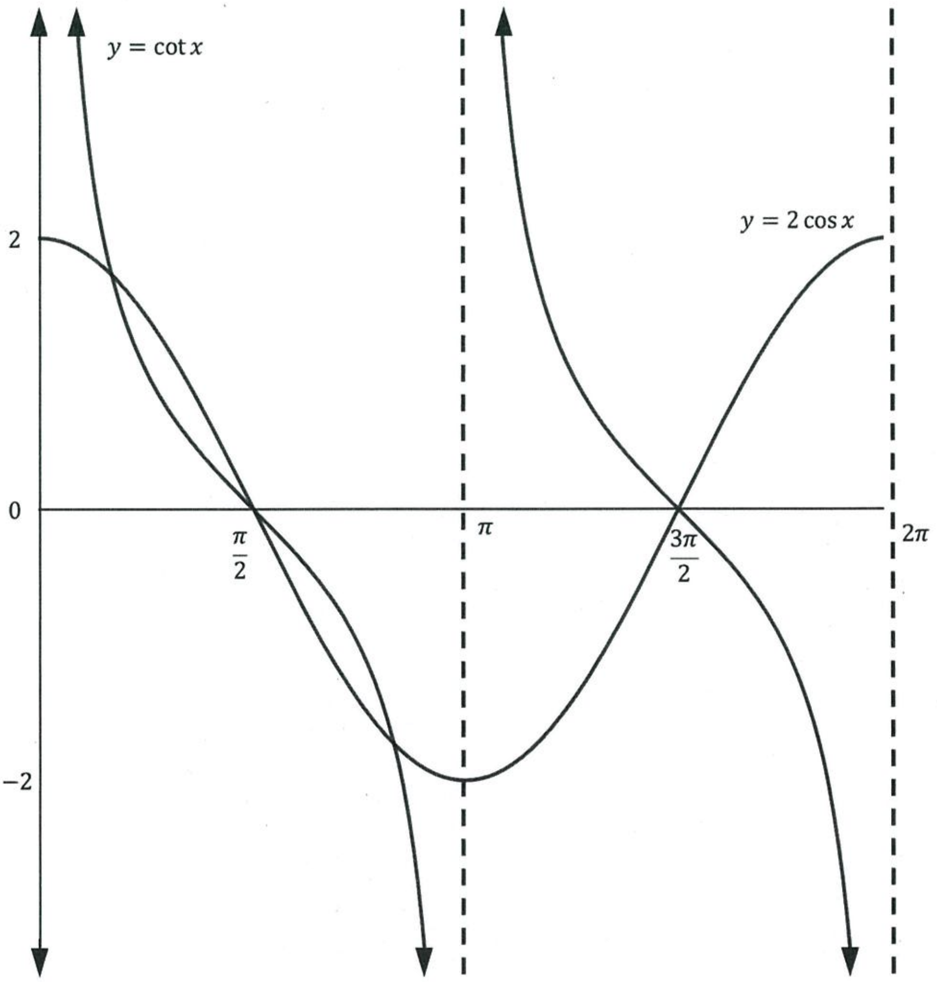
$\frac{d}{dx}xe^x = xe^x + e^x$

$xe^x = -e^x + \frac{d}{dx}xe^x$

$\int xe^x \, dx = -\int e^x \, dx + xe^x$

$= -e^x + xe^x + C$

Question 28 (7 marks)



The diagram shows the graphs of the functions $y = 2\cos x$ and $y = \cot x$ for $0 \le x \le 2\pi$.

(a) Write down the periods of each of the functions $y = 2\cos x$ and $y = \cot x$. **1**

$y = 2\cos x$ $y = \cot x$

Period $= 2\pi$ Period $= \pi$

(b) Show that the x −coordinates of the points of intersections of the two curves are the solutions of the equation $\cos x\,(2\sin x - 1) = 0$ for $0 \le x \le 2\pi$, then solve this equation. **3**

$2\cos x = \cot x$

$2\cos x = \frac{\cos x}{\sin x}$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$ as required

$\therefore \cos x = 0 \qquad \sin x = \frac{1}{2} \qquad 0 \le x \le 2\pi$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{6}$

(c) Find in simplest exact form the total area of the bounded region enclosed by the two curves for $0 \leq x \leq \pi$. **3**

$$\text{Area} = 2 \times \int_{\pi/6}^{\pi/2} 2\cos x - \cot x \, dx$$

$$= 2 \times \int_{\pi/6}^{\pi/2} 2\cos x - \frac{\cos x}{\sin x} \, dx$$

$$= 2 \times \left[2\sin x - \ln \sin x \right]_{\pi/6}^{\pi/2}$$

$$= 2 \times \left[2\sin\frac{\pi}{2} - \ln\sin\frac{\pi}{2} - \left(2\sin\frac{\pi}{6} - \ln\sin\frac{\pi}{6}\right) \right]$$

$$= 2 \times \left[2 - 0 - \left(1 - \ln\frac{1}{2}\right) \right]$$

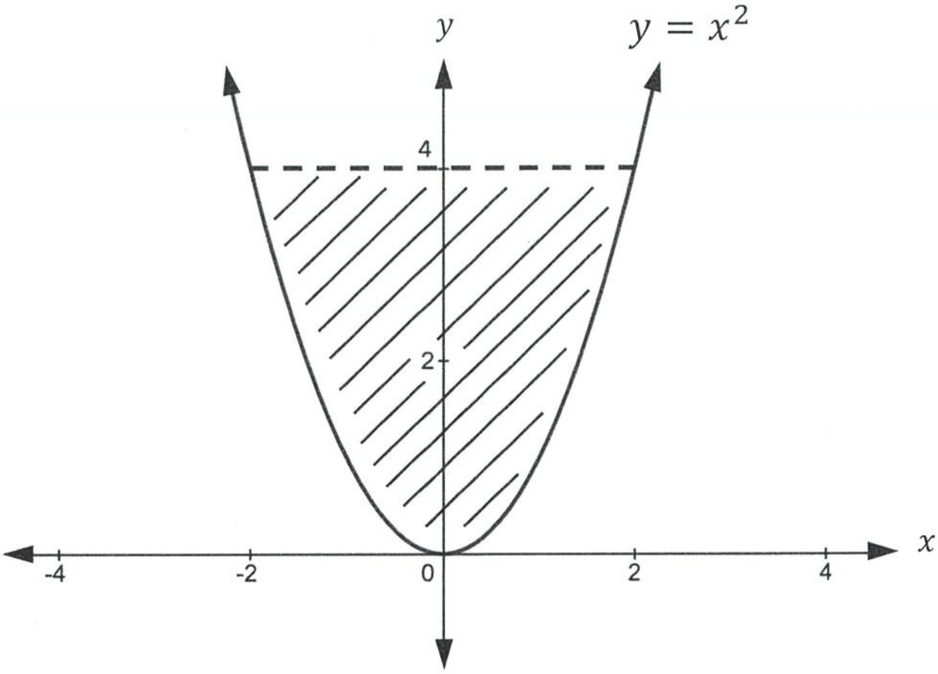
$$= 2\left(2 - 1 + \ln\frac{1}{2}\right)$$

$$= 2(1 - \ln 2)$$

$$= 2 - 2\ln 2 \text{ units squared}$$

Question 29 (3 marks) **3**

Find the area of the shaded region below:



$x = \pm\sqrt{y}$

$\text{Area} = 2 \times \int_0^4 y^{\frac{1}{2}}\, dy$

$= 2 \times \left[\frac{2y^{3/2}}{3} \right]_0^4$

$= 2 \times \frac{2(4)^{3/2}}{3}$

$= \frac{32}{3}$ units squared

Question 30 (4 marks)

A population, P, which is initially 5000, varies according to the formula $P = 5000b^{-\frac{t}{10}}$, where b is a positive constant and t is time in years, $t \geq 0$. 4

The population is 1250 after 20 years. Find the value of t, correct to one decimal place, for which the instantaneous rate of decrease is 30 people per year.

$P = 1250$ when $t = 20$

$1250 = 500b^{-2}$

$\frac{1}{4} = b^{-2}$

$\frac{1}{4} = \frac{1}{b^2}$

$b^2 = 4$

$b = 2 \quad b > 0$ (positive constant)

$\therefore P = 5000 . 2^{-t/10}$

$\frac{dP}{dt} = \ln 2 \times -\frac{1}{10} \times 5000 \times 2^{-t/10}$

$= -500 \ln 2 . 2^{-t/10}$

Find t when $\frac{dP}{dt} = -30$

$-30 = -500 \ln 2 . 2^{-t/10}$

$2^{-t/10} = \frac{3}{50 \ln 2}$

$\ln\left(2^{-t/10}\right) = \ln\left(\frac{3}{50 \ln 2}\right)$

$t = \frac{-10 \ln\left(\frac{3}{50 \ln 2}\right)}{\ln 2}$

$= 35.3$ years

Question 31(5 marks)

(a) Show that $\frac{d}{dx}[\log_4(\tan x)] = \frac{1+\tan^2 x}{\ln 4 \tan x}$. 3

$LHS = \frac{d}{dx} \log_4 \tan x$

$= \frac{d}{dx} \frac{\ln \tan x}{\ln 4}$

$= \frac{1}{\ln 4} \frac{d}{dx} \ln \tan x$

$= \frac{1}{\ln 4} \times \frac{\sec^2 x}{\tan x}$

$= \frac{1+\tan^2 x}{\ln 4 \tan x}$

$= RHS$

(b) Hence, find the angle of inclination to $y = \log_4(\tan x)$ at $x = \frac{\pi}{4}$ to the nearest degree. 2

$m_{tangent} = \frac{1+(\tan\frac{\pi}{4})^2}{\ln 4 \tan\frac{\pi}{4}}$

$= \frac{2}{\ln 4}$

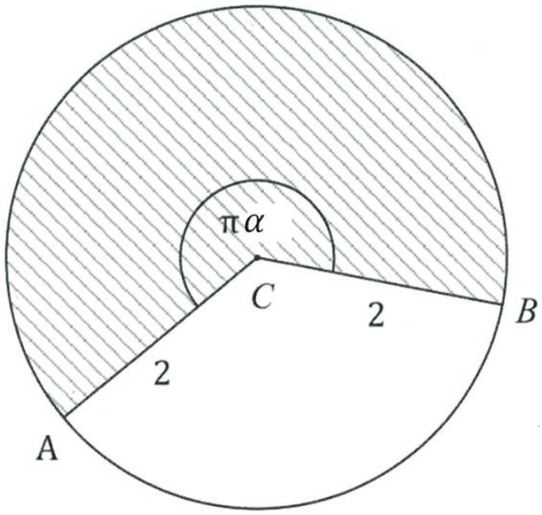
$m = \tan\theta$

$\frac{2}{\ln 4} = \tan\theta$

$\theta = \tan^{-1}\left(\frac{2}{\ln 4}\right)$

$= 55^\circ$

Question 32 (7 marks)



The reflex angle at the centre C of a circle of radius 2cm is $\pi\alpha$ radians, $0 < \alpha < 2$, as shown in the diagram above.

(a) Find the length of the arc of the shaded sector. **1**

$l = r\theta$

$= 2\pi\alpha$

(b) The shaded sector is cut from the circle along the radii CA and CB and folded to make a cone. Find the radius of the cone, in terms of α. **1**

$C = 2\pi r$

$2\pi r = 2\pi\alpha$

$r = \alpha$

(c) Hence, show that the volume of the cone is given by $V = \frac{\pi}{3}\sqrt{4\alpha^4 - \alpha^6}$ **1**

$V = \frac{1}{3}\pi r^2 h \qquad h = \sqrt{l^2 - r^2}$

$V = \frac{1}{3}\pi\alpha^2\sqrt{4-\alpha^2}$

$= \frac{\pi}{3}\sqrt{a^4(4-\alpha^2)}$

$= \frac{\pi}{3}\sqrt{4\alpha^4-\alpha^6}$ as required

(d) Find the value of α, to 2 decimal places, for which the cone is maximised. **4**

$V = \frac{\pi}{3}\sqrt{4\alpha^4-\alpha^6}$

$V = \frac{\pi}{3}(4\alpha^4-\alpha^6)^{\frac{1}{2}}$

$\frac{dV}{d\alpha} = \frac{\pi}{6}(4\alpha^4-\alpha^6)^{-\frac{1}{2}}(16\alpha^3-6\alpha^5)$

let $\frac{dV}{d\alpha} = 0$

$\frac{\pi(16\alpha^3-6\alpha^5)}{6\sqrt{4\alpha^4-\alpha^6}} = 0$

END OF PAPER

Extra writing space.
If you use this space, clearly indicate which question you are answering.

$\therefore 16\alpha^3 - 6\alpha^5 = 0$

$2\alpha^3(8 - 3\alpha^2) = 0$

$\alpha \neq 0 \quad \alpha = 1.63 \quad \alpha > 0$

α	1	1.63	1.8
$dV/d\alpha$	3.02	0	-3.72
Slope	/	—	\

$\therefore$ it is a maximum

when $\alpha = 1.63$ volume is maximised